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ON STABILITY OF SECONDARY FLOWS OF A VISCOUS FLUID IN UNBOUNDED SPACE

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A. A. NEPOMNIASHCHII

(Perm')

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The secondary motions arising as the result of instability in a plane parallel flow of a viscous incompressible fluid in an unbounded space is considered. The flow has a sinusoidal velocity profile at the value of Reynolds nùmber slightly exceeding its critical value over the whole interval $0 < \alpha < \alpha_m$ of the wave numbers (α_m is the wave number of a neutral perturbation of a plane-parallel flow).

The stability of the secondary motions relative to the perturbations which upset the periodic character of the motion is investigated. The method of expansion in α_m, applied previously in [1] to investigate the stability of the wave motions in a film of a viscous fluid running down an inclined plane is used. It is shown that all secondary spatially periodic motions are unstable. The linear theory of stability of a plane parallel flow is formulated for a sinusoidal [2] and arbitrary [3] periodic velocity profile. Bifurcation methods are used to prove the existence of secondary motions. Secondary self-oscillatory modes are constructed and their stability with respect to the perturbations of the same periodicity [4, 5] is investigated. The secondary motions and their stability were also investigated taking a finite number of the Fourier components into account [6, 7]. The higher harmonics for a given value of the Reynolds number can be neglected only in the case of motions the wave numbers α of which differ little from α_m.

1. Let a viscous incompressible fluid of density ρ and kinematic viscosity ν move under the action of an external force parallel to the x -axis and varying in the direction of the y -axis in accordance with the law

$$F = A \sin (y / l)$$

Choosing l, ν / l and $\rho\nu^2 / l^2$ as the unit length, velocity and pressure respectively, we can write the equations of motion in the dimensionless form as follows :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \Delta \mathbf{v} + R \sin y \mathbf{i}, \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

where $\mathbf{i}$ is the unit vector in the direction of the x-axis and $R = Al^3 / \rho\nu^2$ is the Reynolds number. The motion is assumed plane $(v_z = 0)$. We shall use the notation $u = v_x$, $v = v_y$. We also require that the mean flow of the fluid is zero (condition of closure)

$$\lim_{L\to\infty} \frac{1}{L} \int_{-L}^{L} u(x, y)\, dy = \lim_{L\to\infty} \frac{1}{L} \int_{-L}^{L} v(x, y)\, dx = 0 \tag{1.2}$$

and, that the functions $\mathbf{v}$ and ∇p are bounded. The system (1. 1) always has a solution which satisfies the conditions (1. 2) and the condition of boundedness

$$\mathbf{v} = \mathbf{V} = (R \sin y, 0), \quad p = P = \text{const} \tag{1.3}$$

which corresponds to a plane parallel motion. The motion (1. 3) becomes unstable with increasing R, under the long-wave perturbations [2]. Expansion in terms of the wave number α yields the following expression for the perturbation decay increments λ :

$$\lambda = (1 - R^2 / 2)\, \alpha^2 + R^2 (1 + R^2 / 4)\, \alpha^4 + O(\alpha^6) \qquad (1.4)$$

The above expression shows that when $R > R_* = \sqrt{2}$, the perturbations with wave numbers lying in the interval $0 < \alpha < \alpha_m$ increase. The relation connecting α_m and R is given by

$$R = R_* [1 + 3\alpha_m^2 / 2 + O(\alpha_m^4)] \qquad (1.5)$$

From (1. 4) and (1. 5) it follows that for small α_m and $0 < \alpha < \alpha_m$.

$$\lambda(\alpha) = O_1(\alpha_m^4)$$

2. Let us now consider the motions which are not plane parallel. We shall assume α_m to be small since $R - R_*$ is small, and use it as a small parameter. Taking due notice of the estimate given above for λ, it is natural to introduce the variables $X = \alpha_m x$, and $T = \alpha_m^4 t$ for the motions with wave numbers $0 < \alpha < \alpha_m$, $\alpha \Rightarrow O(\alpha_m)$ We shall therefore set

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}, \quad p' = p - P$$

and construct the solutions v' and p' in the form of a series in α_m.

It can be shown that for the steady secondary motions the expansions for the vel - ocity components u' and v' commence with the first power of α_m [7] , and those for p' with the second power of α_m . Let is introduce the functions u v and p for which the expansions begin with the zero power of α_m

$$u = \sum_{n=0}^{\infty} u_n \alpha_m^n, \quad v = \sum_{n=0}^{\infty} v_n \alpha_m^n, \quad p = \sum_{n=0}^{\infty} p_n \alpha_m^n \qquad (2.1)$$

$$(u' = \alpha_m u,\ v' = \alpha_m v,\ p' = \alpha_m^2 p)$$

The relation connecting α_m and R is given by

$$R = \sum_{n=0}^{\infty} R_{2n} \alpha_m^{2n} \qquad (2.2)$$

Equations (1. 1) in the above notation become

$$\frac{\partial v}{\partial y} = -\alpha_m \frac{\partial u}{\partial X} \qquad (2.3)$$

$$\frac{\partial^2 u}{\partial y^2} = R \cos y v + \alpha_m \left(R \sin y \frac{\partial u}{\partial X} + v \frac{\partial u}{\partial y}\right) + \alpha_m^2 \left(u \frac{\partial u}{\partial X} - \frac{\partial^2 u}{\partial X^2} + \frac{\partial p}{\partial X}\right) + \alpha_m^4 \frac{\partial u}{\partial T}$$

$$\frac{\partial p}{\partial y} = -\frac{\partial^2 u}{\partial X \partial y} - R \sin y \frac{\partial v}{\partial X} - v \frac{\partial v}{\partial y} + \alpha_m \left(-u \frac{\partial v}{\partial X} + \frac{\partial^2 v}{\partial X^2}\right) - \alpha_m^3 \frac{\partial v}{\partial T}$$

where the last equation is derived from an equation for the y-component of the velocity with the equation of continuity taken into account.

Let us substitute the expansions (2. 1) and (2. 2) into (2. 3) and equate the terms of like power in α_m. The zero order equations yield

$$\frac{\partial v_0}{\partial y}=0,\quad \frac{\partial^2 u_0}{\partial y^2}=R_0\cos y v_0,\quad \frac{\partial r_0}{\partial y}=-\frac{\partial^2 u_0}{\partial X\partial y}-R_0\sin y\frac{\partial v_0}{\partial X}$$

and from these we obtain

$$v_0=v_0(X,\ T),\quad u_0=-R_0\cos y v_0+u_0^{(1)}(X,\ T)$$

$$p_0=2R_0\cos y\frac{\partial v_0}{\partial X}+p_0^{(1)}(X,T)$$

The conditions of solvability of the equations of higher orders yield the functions $u_0^{(1)}$ and $p_0^{(1)}$, and a closed nonlinear equation for the function $v_0\ (X,\ T)$.
From the first order equations we obtain

$$v_1=R_0\sin y\frac{\partial v_0}{\partial X}+v_1^{(1)}(X,T)$$

$$u_1=-R_0\sin y v_0^2-R_0\cos y v_1^{(1)}+u_1^{(1)}(X,T)$$

$$p_1=R_0\sin y\frac{\partial}{\partial X}(v_0^2)+\frac{1}{2}\sin 2y\frac{\partial^2 v_0}{\partial X^2}+$$
$$2R_0\cos y\frac{\partial v_1^{(1)}}{\partial X}+p_1^{(1)}(X,T)$$

Moreover, from the conditions of solvability of the equations for v_1 and p_1 with (1. 2) taken into account and from the condition of boundedness follows

$$u_0^{(1)}=0,\quad R_0=\sqrt{2}$$

For the sake of brevity, we shall denote by repeated dots the terms related to the sol - utions of homogeneous equations $v_n^{(1)}\ (X,\ T), u_n^{(1)}\ (X,\ T)$ and $p_n^{(1)}\ (X,\ T),\ \ n\geqslant 1$, as these functions do not appear in the final equation. In the second order we have

$$v_2=-R_0\cos y\frac{\partial}{\partial X}(v_0^2)+\ldots$$

$$u_2=-R_2\cos y v_0-R_0\cos y\left(3\frac{\partial^2 v_0}{\partial X^2}-v_0^3\right)+\ldots$$

(the expression for p_2 is not used) . The condition of solvability of the equation for u_2 yields

$$p_0^{(1)}=v_0^2+C$$

where C is an arbitrary constant. Finally, in the third order the condition of solv - ability of the equation for p_3 we obtain the closed nonlinear equation for the function v_0

$$\frac{\partial v_0}{\partial T}+3\frac{\partial^4 v_0}{\partial X^4}+R_2R_0\frac{\partial^2 v_0}{\partial X^2}-\frac{2}{3}\frac{\partial^2}{\partial X^2}(v_0^3)=0 \qquad (2.4)$$

The equation which can be obtained by linearizing (2. 4) must, by virtue of α_m being defined as the wave number of a neutral perturbation, have a time-independent solution $v_0=\exp\ (iX)$. This implies that $R_2={}^3/_2\sqrt{2}$ which agrees with (1. 5). Transforming the scale we can reduce (2. 4) to the form

$$\frac{\partial V}{\partial\tau}+\frac{\partial^4 V}{\partial X^4}+\frac{\partial^2 V}{\partial X^2}-\frac{\partial^2}{\partial X^2}(V^3)=0 \qquad (2.5)$$
$$(v_0={}^3/_2\sqrt{2}V,\ \tau=3T)$$

We note that (2. 5) can also be derived using the stream function equation instead of the system (2. 3), but in this case the final equation will be of higher order in α_m.

3. Let us consider steady, secondary periodic motions described by the equation

$$d^2/dX^2\,(d^2V/dX^2 + V - V^3) = 0 \tag{3.1}$$
$$V\,(X + 2\pi\,/\,\alpha_1) = V\,(X), \quad 0 < \alpha_1 < 1$$

where α_1 denotes the ratio of the wave number α of the secondary motion, to α_m. Taking the requirement of the boundedness of V and the conditions (1. 2) into account, we find that a suitable choice of the reference point in X yields a solution of the form

$$V = \sqrt{\frac{2k^2}{1+k^2}}\,\mathrm{sn}\,\frac{X}{\sqrt{1+k^2}} \tag{3.2}$$

where k is the modulus of the elliptic function. The condition of periodicity gives the following relation connecting k and α_1 :

$$\alpha_1 = \frac{\pi}{2K\,(k)\,\sqrt{1+k^2}} \tag{3.3}$$

where $K\,(k)$ is the complete elliptic integral, $k \to 0$ as $\alpha_1 \to 1$ and $k \to 1$ as $\alpha_1 \to 0$. From (3. 2) it follows that the higher order Fourier components in X can be neglected when α_1 is nearly equal to unity.

Let us now investigate the stability of the solution (3. 2). For perturbations $W\,(X)\,\exp\,(-\lambda\tau)$ superimposed on the solution (3. 2) we have

$$-\lambda W + \frac{d^4W}{dX^4} + \frac{d^2\,[W\,(1 - 3V^2)]}{dX^2} = 0 \tag{3.4}$$

The eigenvalues λ can be found from the condition that W remains bounded as $X \to \pm\infty$. Since V^2 is a $\pi\,/\,\alpha_1$-periodic function, a restricted solution of (3. 4) can be written in the form

$$W\,(X) = U\,(X)\,\exp\,(i\beta X) \tag{3.5}$$

Here $U\,(X)$ is a $\pi\,/\,\alpha_1$-periodic function and β is a real number defined with the accuracy to within an integer multiple of $2\alpha_1$, so that we can assume that $-\alpha_1 \leqslant \beta \leqslant \alpha_1$. The function $U\,(X)$ satisfies the equation and condition of periodicity

$$-\lambda U + \left(\frac{d}{dX} + i\beta\right)^4 U + \left(\frac{d}{dX} + i\beta\right)^2 [(1 - 3V^2)\,U] = 0 \tag{3.6}$$
$$U\,(X + \pi\,/\,\alpha_1) = U\,(X)$$

Let us consider the perturbations with small β. We shall seek a solution of (3. 6) and the quantity λ in the form of a series in β,

$$U = \sum_{n=0}^{\infty} U_n\beta^n, \quad \lambda = \sum_{n=0}^{\infty} \lambda_n\beta^n$$

In the zeroth order we have

$$-\lambda_0 U_0 + U_0^{\mathrm{IV}} + [(1 - 3V^2)U_0]'' = 0 \tag{3.7}$$
$$U_0\,(X + \pi\,/\,\alpha_1) = U_0\,(X)$$

where a prime denotes differentiation with respect to X. Equation (3.7) has a solution for $\lambda_0 = 0$, of the form

$$U_0 = k'^2 + 2k^2 \mathrm{cn}^2 \frac{X}{\sqrt{1+k^2}}, \quad k'^2 = 1 - k^2 \tag{3.8}$$

Consider the following orders in β for a perturbation with $\lambda_0 = 0$. In the first order in β we have

$$U_1^{\mathrm{IV}} + [(1 - 3V^2)U_1]'' = \lambda_1 U_0 - 4iU_0''' - 2i\,[(1 - 3V^2)U_0]'$$

$$U_1\,(X + \pi / \alpha_1) = U_1\,(X)$$

The left-hand side of the equation contains a complete derivative of the periodic function. Obviously the condition of solvability has the form

$$\lambda_1 \langle U_0 \rangle = 0$$

$$\left(\langle f \rangle = \frac{\alpha_1}{\pi} \int\limits_0^{\pi/\alpha_1} f(X)\,dX\right)$$

Since $\langle U_0 \rangle \neq 0$, we have $\lambda_1 = 0$. In the second order in β we have

$$U_2'^{v} + [(1 - 3V^2)U_2]'' = \lambda_2 U_0 - 4iU_1''' - 2i\,[(1 - 3V^2)U_1]'$$
$$+ 6U_0'' + (1 - 3V^2)U_0, \quad U_0\,(X + \pi / \alpha_1) = U_0\,(X)$$

The condition of solvability yields

$$\lambda_2 = -\,\langle(1 - 3V^2)U_0\rangle \,/\, \langle U_0 \rangle$$

Substituting (3.2) and (3.8) in the above expression, we obtain

$$\lambda_2 = -k'^4 \,/\, (2E\,(k) \,/\, K\,(k) - k'^2)(1 + k^2), \quad k'^2 = 1 - k^2$$

where $K\,(k)$ and $E\,(k)$ are complete elliptic integrals of the first and second kind. It can be shown that λ_2 is negative in the whole region $0 < k < 1$, so that all periodic secondary motions exhibit a mode of perturbation the decay decrement of which is negative for small but finite values of β.

Thus we see that although the secondary spatially periodic motions are stable with respect to the perturbations the period of which coincides with the period of the motion under consideration [5], they are unstable with respect to the perturbations of more general type. In contrast with the secondary motions in a layer with rigid boundaries [8] and a motion with a free surface [1] for which an interval of stability with respect to plane perturbations exists, the secondary spatially periodic motions arising as the result of instability in a plane parallel flow with a sinusoidal velocity profile in an unbounded region, are unstable irrespective of the values of their periods.

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CONVECTIVE DIFFUSION TO A REACTING RIGID SPHERE IN STOKES FLOW

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Iu. A. SERGEEV
(Moscow)

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The problem of convective diffusion to a reacting rigid sphere was solved earlier in [1] for small values of Péclet and Reynolds numbers and finite reaction velocities, using the method of matched asymptotic expansions. In the present paper the problem of diffusion to a rigid sphere in a Stokes flow at finite velocities of the first order chemical reaction at the sphere surface is solved for large values of the Péclet number. The method of solution is similar to that used in [2] in the problem of convective diffusion to a reacting flat plate in a longitudinal flow of a viscous fluid.

We consider a convective diffusion of material to a rigid sphere in a Stokes flow of a viscous incompressible fluid the speed of which, away from the sphere is U. We assume that the Péclet numbers $P = aU / D$ (where a is the radius of the sphere and D is the diffusion coefficient of the material in the flow) are large. A first order chemical reaction with the velocity constant k takes place at the surface of the sphere. The process of convective diffusion at large Péclet numbers is described by the boundary layer diffusion equation which in the spherical (r, θ) -coordinate system with the origin at the center of the sphere and the polar axis pointing in the direction opposite to the direction of flow at infinity, has the form

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r} \frac{\partial c}{\partial \theta} = D \frac{\partial^2 c}{\partial r^2} \tag{1}$$

Here v_r and v_θ are the radial and angular velocity components in the spherical